Vanishing theorems for associative submanifolds

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Abstract

Let M^7 a manifold with holonomy in G_2, and Y^3 an associative submanifold with boundary in a coassociative submanifold. In [5], the authors proved that $\mathcal{M}_{X,Y}$, the moduli space of its associative deformations with boundary in the fixed X, has finite virtual dimension. Using Bochner's technique, we give a vanishing theorem that forces $\mathcal{M}_{X,Y}$ to be locally smooth.

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1 Introduction

Let M a 7-dimensional riemannian manifold with holonomy included in G_2. In this case by parallel transport, M supports a non degenerate harmonic 3-form ϕ with $\nabla\phi = 0$. Moreover, M benefits a vector product $\times$ defined by

$$< u \times v, w >= \phi(u, v, w),$$

even if to be spin is enough for the existence of this product. A 3-dimensional submanifold Y is said *associative* if its tangent bundle is stable under the vector product. In other terms, ϕ restricted to Y is a volume form. Likely, a 4-dimensional submanifold X is *coassociative* if the fibers of its normal bundle are associative, or equivalently, $\phi_{|TX}$ vanishes. We refer to the abundant literature on this subject, see [6] or [5] for a summary with our notations.

The closed case. It is known from [10] that the deformation of an associative submanifold Y without boundary is an elliptic problem, and hence is of vanishing index. In general, the situation is obstructed. For instance, consider the flat torus $\mathbb{T}^3 \times \{pt\}$ in the flat torus $\mathbb{T}^7 = \mathbb{T}^3 \times \mathbb{T}^4$. This is an associative submanifold, and its moduli space $\mathcal{M}_{\mathbb{T}^3\times\{pt\}}$ of associative deformations contains at least the 4-dimensional $\mathbb{T}^4$.

A natural question is to find conditions which force the moduli space to be smooth, or in other terms, which force the cokernel of the problem to vanish. For the closed case, Abkulut and Salur [2] allow a certain freedom for the connection on the normal bundle, the definition of associativity and genericity. But examples are often non generic, and we would like to get a condition that is not a perturbative one. For holomorphic curves in dimension 4, there are topological conditions on the degree of the normal bundle which imply smoothness of the moduli space of complex deformations, see for example [7]. The main point for this is

that holomorphic curves intersect positively. In our case, there is no such phenomenon.

In [10], page 30, McLean gives an example of an isolated associative submanifold. Since this was the start point of our work, we recall it. In [4], Bryant and Salamon constructed a metric of holonomy G_2 on the spin bundle $S^3 \times \mathbb{R}^4$ of the round 3-sphere. In this case, the basis $S^3 \times \{0\}$ is associative, the normal bundle is just the spin bundle of S^3, and the operator related to the associative deformations of S^3 is just the Dirac operator. By the famous theorem of Lichnerowicz [9], there are no non trivial harmonic spinors on S^3 for metric reasons (precisely, the riemaniann scalar curvature is positive), so the sphere is isolated as an associative submanifold.

Minimal submanifolds. Recall that in manifolds with holonomy in G_2, associative submanifolds are minimal (the condition $d\phi = 0$ is enough). In [11], Simons gives a metric condition for a minimal submanifold to be stable, i.e isolated. For this, he introduce the following operator, a sort of partial Ricci operator :

Definition 1.1 *Let (M, g) a riemaniann manifold, Y^p a submanifold in M and ν its normal bundle. Choose $\{e_1, \cdots e_p\}$ a local orthonormal frame field of TY, and define the 0-order operator*

$$\begin{array}{rcl} \mathcal{R} : \Gamma(Y, \nu) & \longrightarrow & \Gamma(Y, \nu) \\ \psi & \mapsto & \pi_\nu \sum_{i=1}^{p} R(e_i, \psi) e_i, \end{array}$$

where R is the curvature tensor on M and π_ν the orthogonal projection on ν.

Fact. The definition is independant of the choosen oriented orthonormal frame, and $\mathcal{R}$ is symmetric.

He introduces another operator $\mathcal{A}$ related to the second fondamental form of Y :

Definition 1.2 *Let SY the bundle over Y whose fibre at a point y is the space of symmetric endomorphisms of T_yY, and $A \in Hom(\nu, SY)$ the second fundamental form defined by*

$$A(\phi)(u) = -\nabla_u^\top \phi,$$

where $u \in TY$, $\phi \in \nu$, and $\nabla^\top$ is the projection on TY of the ambient Levi-Civita connection. Consider the operator

$$\begin{array}{rcl} \mathcal{A} : \Gamma(Y, \nu) & \longrightarrow & \Gamma(Y, \nu) \\ \psi & \mapsto & A^t \circ A(\psi), \end{array}$$

where A^t is the transpose of A.

Fact. This is a symmetric positive 0-th order operator. Moreover, it vanishes if Y is totally geodesic.

Using both operators, Simons gives a sufficient condition for a minimal submanifold to be stable :

Theorem 1.1 ([11]) *Let Y a minimal submanifold in M, and suppose that $\mathcal{R} - \mathcal{A}$ is positive. Then Y cannot be deformed as a minimal submanifold.*

Bochner technique. If Y is an associative submanifold in M, we will recall that there is an operator D acting on the normal vector fields of Y, such that its kernel can be identified with the infinitesimal associative deformations of Y. We will compute D^2 to use the Bochner method, and get vanishing theorems. For this, we introduce the normal equivalent of the invariant second derivative. More precisely, for every local vector fields v and w in $\Gamma(Y, TY)$, let

$$\nabla^{\perp 2}_{v,w} = \nabla^{\perp}_{v}\nabla^{\perp}_{w} - \nabla^{\perp}_{\nabla^{\top}_{v} w},$$

acting on $\Gamma(Y, \nu)$. It is straightforward to see that it is tensorial in v and w. Moreover, define the equivalent of the connection laplacian :

$$\nabla^{\perp *}\nabla^{\perp} = -\text{trace}\ (\nabla^{\perp 2}) = -\sum_i \nabla^{\perp 2}_{e_i,e_i}.$$

Theorem 1.2 *For Y an associative submanifold, $D^2 = \nabla^{\perp *}\nabla^{\perp} + \mathcal{R} - \mathcal{A}$.*

Remark. In fact this shows that for closed submanifolds, associativity does not give more conditions than the one for minimal submanifods, as long as we use this method.

The case with boundary. In [5], the authors proved that the deformation of an associative submanifold Y with boundary in a coassociative submanifold X is an elliptic problem of finite index. Moreover, they gave the value of this index in terms of a certain Cauchy-Riemann operator related to the complex geometry of the boundary. We sum up in the following the principal results of the paper :

Theorem 1.3 ([5]) *Let ν_X the normal complementary of $T\partial Y$ in $TX_{|\partial Y}$, and n the inner unit vector normal to ∂Y in Y. Then the bundle ν_X is a subbundle of $\nu_{|\partial Y}$ and is stable under the left action by n under $\times$, as well as the orthogonal complement μ_X of ν_X in ν. Viewing $T\partial Y$, ν_X and μ_X as $n\times$-complex line bundles, we have $\mu_X^* \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$. Besides, the problem of the associative deformations of Y with boundary in X is elliptic and of index*

$$\text{index}\,(Y, X) = \text{index}\,\overline{\partial}_{\nu_X} = c_1(\nu_X) + 1 - g,$$

where g is the genus of ∂Y.

In this context, we introduce a new geometric object that is related to the geometry on the boundary :

Proposition 1.4 *Choose $\{v, w = n \times v\}$ a local orthonormal frame for $T\partial Y$. Let L a real plane subbundle of ν invariant by the action of $n\times$. We define*

$$\begin{array}{rccl} \mathcal{D}_L : \Gamma(\partial Y, L) & \longrightarrow & \Gamma(\partial Y, L) \\ \phi & \mapsto & \pi_L(v \times \nabla^{\perp}_{w}\phi - w \times \nabla^{\perp}_{v}\phi), \end{array}$$

where π_L is the orthogonal projection on L and $\nabla^{\perp}$ the normal connection on ν induced by the Levi-Civita connection ∇ on M. Then $\mathcal{D}_L$ is independant of the choosen oriented frame, of order 0 and symmetric.

Now, we can express our main theorem :

Theorem 1.5 *Let Y an associative submanifold of a G_2-manifold M with boundary in a coassociative X. If $\mathcal{D}_{\mu_X}$ and $\mathcal{R} - \mathcal{A}$ are positive, the moduli space $\mathcal{M}_{Y,X}$ is locally smooth and of dimension given by the virtual one* $\text{index}\,(Y, X)$.

When $M = \mathbb{R}^7$, we get the following very explicit example considered in [5]. Take a ball Y in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$, with real analytic boundary, and choose e any constant vector field in $\nu = Y \times \{0\} \oplus \mathbb{R}^4$. By [6], there is a unique local coassociative X_e containing $\partial Y \times \mathbb{R}e$, such that

$$TX_{|\partial Y} = T\partial Y \oplus \nu_X = T\partial Y \oplus \mathrm{Vect}\ (e, n \times e).$$

Of course, the translation in the e-direction gives associative deformations of Y with boundary in X_e. The next corollary shows that under a simple metric condition, this is the only way to deform Y :

Corollary 1.6 *If Y is a strictly convex ball in $\mathbb{R}^3$, then $\mathcal{M}_{Y,X_e} = \mathbb{R}$.*

The Calabi-Yau extension. Let (N, J, Ω, ω) a Calabi-Yau 6-dimensional manifold, where J is an integrable complex stucture, Ω a non vanishing holomorphic 3-form and ω a Kähler form. Here we allow holonomies which are only subgroups of $SU(3)$. Then $M = N \times S^1$ is a manifold with holonomy in $SU(3) \subset G_2$. The associated calibration 3-form is given by

$$\phi = \omega \wedge dt + \Re\Omega.$$

Recall that a special lagrangian in N is 3-dimensional submanifold L in N satisfying both conditions $\omega_{|TL} = 0$ and $\Im\Omega_{|TL} = 0$. We know from [10] that $\mathcal{M}_L$ the moduli space of special lagrangian deformations of L is smooth and of dimension $b^1(L)$. Now every product $Y = L \times \{pt\}$ of a special lagrangian and a point is an associative submanifold of M.

If Σ is a complex surface of N, then $X = \Sigma \times \{pt\}$ is a coassociative submanifold of M. Consider the problem of associative deformations of $Y = L \times \{pt\}$ with boundary in X :

Theorem 1.7 *Let L a special lagrangian submanifold in a 6-dimensional Calabi-Yau N, such that L has boundary in a complex surface Σ. Let $Y = L \times \{t_0\}$ in $N \times S^1$ and $X = \Sigma \times \{t_0\}$. If the Ricci curvature of L is positive, and the boundary of L has positive mean curvature in L, then $\mathcal{M}_{Y,X}$ is locally smooth and has dimension g, where g is the genus of ∂L.*

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2 Closed associative submanifolds

2.1 The operator D

We begin with the version of McLean's theorem proposed by Akbulut and Salur. We will give below a new proof of this result.

Theorem 2.1 ([10],[1]) *Let Y an associative submanifold of a riemannian manifold M with G_2-holonomy, and ν its normal bundle. Then the tangent space of its associative deformations is the kernel of the operator*

$$\begin{aligned} D : \Gamma(Y,\nu) &\longrightarrow \Gamma(Y,\nu), \\ \psi &\mapsto \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i}\psi, \end{aligned} \qquad (1)$$

where $\nabla^\perp$ is the connection on ν induced by the Levi-Civita connection ∇ on M.

As the proof of theorem 1.2 is pretty technical, we refer to the last section for it.

2.2 The implicit function machinery

We give now the analytic elements needed for the clear definition of our moduli spaces and their smoothness. We define the Banach space $\mathcal{E}$ of (not necessarily associative) embedding of a 3-manifold in M, and a function F that measures the lack of associativity. Then we linearise F, and identify the tangent space of the moduli space of associative deformations, a priori weak, with its kernel. Since the derivative of F is elliptic, those deformations are in fact smooth, and by the implicit function theorem, this moduli space is smooth if the cokernel vanishes.

Firstly, recall the existence on (M, ϕ) of a important object, the 3-form with values in TM defined, for u, v, $w \in TM$ by :

$$\chi(u, v, w) = -u \times (v \times w) - < u, v > w + < v, w > v. \tag{2}$$

It is easy to check [1] that $\chi(u, v, w)$ is orthogonal to the 3-plane $u \wedge v \wedge w$. Besides, if Y is a 3-dimensional submanifold in (M, ϕ), then $\chi_{|TY} = 0$ if and only if Y is associative.

For the future computations, we will use the the following usefull formula [6] :

$$< \chi(u, v, w), \eta >= *\phi(u, v, w, \eta),$$

where $*$ is the Hodge star, and $\eta \in TM$. So

$$\chi = \sum_k \eta_k \lrcorner * \phi \otimes \eta_k, \tag{3}$$

where $(\eta_k)_{k=1,2,\cdots 7}$ is an orthonormal basis of the tangent space of M.

Now, as in [10], we use this characterization to study the moduli space of associative deformations of an associative Y.

Proof of theorem 2.1. Let (Y, g) any riemanian 3-manifod. For every embedding $f : Y \to M$, define

$$F(f) = f^*\chi(\omega) \in \Gamma(Y, f_*TM),$$

where ω is the volume form on Y. Then $f(Y)$ is associative if and only if $F(f)$ vanishes. Consider a path of embeddings $(f_t)_{t\in[0,1]}$. After a reparametrization of Y, we can suppose that

$$s = \frac{df_t}{dt}_{|t=0} \in \Gamma(Y, \nu_{f_0}),$$

where ν_f is the normal bundle over $f(Y)$.

Suppose that $f_0(Y)$ is an associative submanifold of M, and that f_0 is the injection of Y in M. In order to derivate the vector-valued form F at f_0, we use the Levi-Civita connection :

$$\nabla_{\frac{\partial}{\partial t}} F(f_t)_{|t=0} = \sum_k \mathcal{L}_s(\eta_k \lrcorner * \phi)(\omega) \otimes \eta_k + (\eta_k \lrcorner * \phi)(\omega) \otimes \nabla_s \eta_k,$$

where $\mathcal{L}_s$ is the Lie derivative in the direction s. The second member vanishes because Y is associative. Thanks to classical riemannian formulas, we compute the first term :

$$\begin{aligned}\mathcal{L}_s(\eta_k \lrcorner * \phi)(\omega) &= (\eta_k \wedge \omega)\lrcorner\, \mathcal{L}_s(*\phi) + ([\eta_k, s] \wedge \omega)\lrcorner * \phi \\ &= \mathcal{L}_s(*\phi)(\eta_k, \omega),\end{aligned}$$

since $(\eta_k \lrcorner * \phi)(\omega) =< \chi(\omega), \eta_k >= 0$. Writing $\omega = e_1 \wedge e_2 \wedge e_3$, where $(e_i)_{i=1,2,3}$ is a local orthonormal frame of TY with $e_3 = e_1 \times e_2$, this is equal to

$$\nabla_s * \phi(\eta_k, \omega) + *\phi(\nabla_{\eta_k} s, \omega) + *\phi(\eta_k, \nabla_{e_1} s, e_2, e_3)) + *\phi(\eta_k, e_1, \nabla_{e_2} s, e_3)) + *\phi(\eta_k, e_1, e_2, \nabla_{e_3} s).$$

The first term vanishes since $*\phi$ is covariantly constant, the second one vanishes too for the former reason, and the third is

$$*\phi(\eta_k, \nabla_{e_1} s, e_2, e_3) = *\phi(\eta_k, \nabla^{\perp}_{e_1} s, e_2, e_3) = - < \nabla^{\perp}_{e_1} s \times (e_2 \times e_3), \eta_k > .$$

Using equation (3) and the relation $e_2 \times e_3 = e_1$ and summing up the two others similar terms, we have :

$$Ds = \nabla_s F = \sum_i e_i \times \nabla^{\perp}_i s. \qquad (4)$$

■

After this linearisation, we come back to the problem of the moduli space.

Proposition 2.2 *Let Y an associative submanifold in M. If the kernel of the operator D given by (1) vanishes, then $\mathcal{M}_Y$ is locally smooth and of vanishing dimension. If the moduli space $\mathcal{M}_Y$ contains a smooth submanifold $\mathcal{M}_k$ of dimension k, and if for every Y' in $\mathcal{M}_k$ the kernel of D at Y' is of dimension k, then $\mathcal{M}_Y = \mathcal{M}_k$.*

Proof. For $kp > 3$, it has a sense to consider the Banach space

$$\mathcal{E} = W^{k,p}(Y, M),$$

with tangent space at f equal to $T_f\mathcal{E} = W^{k,p}(Y, f_*TM)$. Moreover for $(k-r)/3 > 1/p$, then

$$W^{k,p}(Y, M) \subset C^r(Y, M),$$

and so $f \in \mathcal{E}$ is C^1 if $k > 1 + 3/p$. In particular, one can define $\mathcal{F}$ the Banach bundle over $\mathcal{F}$ with fiber

$$\mathcal{F}_f = W^{k-1,p}(Y, \nu_f).$$

It is clear that the operator F extends to a section $F_{k,p}$ of $\mathcal{F}$ over $\mathcal{E}$. We just proved before that F is differentiable, and if $f_0(Y)$ is associative, its derivative at $f_0(Y)$ along a vector field $s \in T_{f_0}\mathcal{E}$ is given by (4).

Now, the operator D has symbol

$$\sigma(\xi) : s \mapsto \sum_i \xi_i s \times e_i. = s \times \xi,$$

which is always inversible for $\xi \in TY \setminus \{0\}$. This proves that D is elliptic. Remark that $\sigma(\xi)^2 s = -|\xi|^2 s$, which is the symbol of the laplacian. Hence $\dim\ker D$ and $\dim \mathrm{coker} D$ are of finite dimension. By the implicit function theorem for Banach bundles, if $\mathrm{coker}D$ vanishes, then $F^{-1}(0)$ is a locally a smooth Banach submanifold of $\mathcal{E}$ of finite dimension equal to $\dim\ker D = \mathrm{index}\, D$, which vanishes because Y is odd-dimensional. More generally, if $\dim \mathrm{coker}D$ is constant on the component of $\mathcal{M}_X$ containing Y, then the moduli space is still smooth of dimension $\dim\ker D$. Lastly, still because of ellipticity, all elements of $\mathcal{M}_X$ are smooth. ■

2.3 A vanishing theorem

We can now formulate the following theorem, which can be deduced from theorem 1.1, since any associative submanifold is minimal.

Theorem 2.3 *Let Y an associative submanifold of a G_2-manifold M. If the spectrum of $\mathcal{R}_\nu = \mathcal{R} - \mathcal{A}$ is positive, then Y is isolated as an associative submanifold.*

For reader's convenience, we give a proof of this result.

Proof. Suppose that we are given a fixed closed associative submanifold Y. The virtual dimension of its moduli space of deformation is vanishing. Consider a section $\psi \in \Gamma(Y, \nu)$. By classical calculations, using normal coordinates, we have

$$\begin{aligned} -\frac{1}{2}\Delta|\psi|^2 &= \sum_i < \nabla_i^\perp \psi, \nabla_i^\perp \psi > + < \psi, \nabla_i^\perp \nabla_i^\perp \psi > \\ &= |\nabla^\perp \psi|^2 - < D^2\psi, \psi > + < \mathcal{R}_\nu \psi, \psi > \end{aligned}$$

by theorem 1.2. Since the laplacian is equal to $-\mathrm{div}(\vec{\nabla})$, its integral over the closed Y vanishes. We get :

$$0 = \int_Y |\nabla^\perp \psi|^2 - < D^2\psi, \psi > + < \mathcal{R}_\nu \psi, \psi > dy. \tag{5}$$

Suppose that we have a section $\psi \in \ker D$. Under the hypothesis that $\mathcal{R}_\nu$ is positive, the last equation implies $\psi = 0$. Hence $\dim \mathrm{coker} D = \dim \ker D = 0$, and by proposition 2.2, $\mathcal{M}_Y$ is locally a smooth manifold of vanishing dimension, and Y is isolated. ■

3 Associative submanifolds with boundary

In this section we extend our result for rigidity in the case of associative submanifolds with boundary in a coassociative submanifold. In this case the index may be not zero, so rigidity transforms into smoothness of the moduli space.

3.1 Implicit function machinery

As before, define the adapted $\mathcal{E}$, for $kp > 3$ and $(k-r)/3 > 1/p$:

$$\mathcal{E}_X = \{f \in W^{k,p}(Y, M),\ f(\partial Y) \subset X\}.$$

This has the following tangent space :

$$T\mathcal{E}_{X,f} = \{s \in W^{k,p}(Y, f^*TM),\ s_{|\partial Y} \in f^*TX\}.$$

As before, we have the map :

$$F : \mathcal{E}_X \to W^{k-1,p}(Y, \nu_{f(Y)}).$$

It is enough to compute the derivative of F_X at an application f_0 where $f_0(Y)$ is an associative submanifold. We suppose as in the closed case that f_0 is an injection $Y \hookrightarrow M$. In this case, lemma 1.3 showed that TX is orthogonal to TY at ∂Y, hence the derivative of F at f_0 is :

$$D : \{s \in W^{k,p}(Y, \nu),\ s_{|\partial Y} \in \nu_X\} \to W^{k-1,p}(Y, \nu).$$

Now, to get some trace properties and use the results of [3], we need to restrict to the Sobolev space $p = 2$. In particular, if $f \in H^s(Y,\nu) = W^{s,2}(Y,\nu)$, then $f_{|\partial Y} \in H^{s-\frac{1}{2}}(Y,\nu)$. By theorem 20.8 of [3], the operator D is Fredholm. In [5], the authors computed its index, given by theorem 1.3.

Notation : For L a subbundle of $\nu_{|\partial Y}$ of real rank equal to two, define

$$\ker(D,L) = \{s \in W^{k,p}(Y,\nu),\ s_{|\partial Y} \in L,\ Ds = 0\}.$$

We will need the usefull

Proposition 3.1 *The operator D is formally self-adjoint, i.e for s and $s' \in \Gamma(Y,\nu)$,*

$$\int_Y < Ds, s' > - < s, Ds' > dy = -\int_{\partial Y} < n \times s, s' > d\sigma, \tag{6}$$

where $d\sigma$ is the volume induced by the restriction of g on the boundary, and n is the normal inner unit vector of ∂Y. Moreover, $\operatorname{coker}(D,\nu_X) = \ker(D,\mu_X)$.

Proof. The proof of the firs assertion is *mutatis mutandis* the classical one for the classical Dirac operator, see proposition 3.4 in [3] for example. For the reader's convenience we give a proof of this.

$$\begin{aligned} < Ds, s' > &= < \sum_i e_i \times \nabla_i^\perp s, s' >= -\sum_i < \nabla_i^\perp s, e_i \times s' > \\ &= -\sum_i d_{e_i} < s, e_i \times s' > + < s, \nabla_i^\perp (e_i \times s') > \\ &= -\sum_i d_{e_i} < s, e_i \times s' > + < s, \nabla_i^\top e_i \times s' + e_i \times \nabla_i^\perp s' > . \end{aligned}$$

By a classical trick, define the vector field $X \in \Gamma(Y, TY)$ by

$$< X, w >= - < s, w \times s' > \ \forall w \in TY.$$

Note that the first product the one of TY, and the second one the one of ν. Now

$$\begin{aligned} -\sum_i d_{e_i} < s, e_i \times s' > &= \sum_i d_{e_i} < X, e_i > \\ &= \sum_i < \nabla_i^\top X, e_i > + < X, \nabla_i^\top e_i > \\ &= \sum_i \operatorname{div} X - < s, \nabla_i^\top e_i \times s' > . \end{aligned}$$

By Stokes we get

$$\begin{aligned} \int_Y < Ds, s' > dy &= \int_{\partial Y} < X, -n > d\sigma + \int_Y < s, Ds' > dy \\ &= \int_{\partial Y} < s, n \times s' > d\sigma + \int_Y < s, Ds' > dy, \end{aligned}$$

which is what we wanted. Now, let $s' \in \Gamma(Y,\nu)$ lying in $\operatorname{coker}(D,\nu_X)$. This is equivalent to say that for every $s \in \Gamma(Y,\nu_X)$, we have $\displaystyle\int_Y < Ds, s' > dy = 0$. By the former result, we see that this equivalent to

$$\int_Y < s, Ds' > + \int_{\partial Y} < n \times s, s' >= 0.$$

This clearly implies that $Ds' = 0$, and $s'_{|\partial Y}(\nu) \perp \nu_X$, because ν_X is invariant under the action of $n\times$. So $s'_{|\partial Y} \in \mu_X$, and $s' \in \ker(D, \mu_X)$. The inverse inclusion holds too by similar reasons. ■

3.2 Vanishing theorem

Proof of theorem 1.5. In order to get some smooth moduli spaces in the case with boundary, we want to prove that $\mathrm{coker}(D, \nu_X) = \ker(D, \mu_X)$ is trivial or has constant rank. So let $\psi \in \ker(D, \mu_X)$. The boundary changes the integration (5), because the divergence has to be considered :

$$\int_Y |\nabla^\perp \psi|^2 + < \mathcal{R}_\nu \psi, \psi > dy = \frac{1}{2} \int_Y \mathrm{div}\ \vec{\nabla} |\psi|^2 dy. \tag{7}$$

By Stokes, the last is equal to

$$-\frac{1}{2} \int_{\partial Y} d|\psi|^2(n) d\sigma = -\int_{\partial Y} < \nabla_n^\perp \psi, \psi > d\sigma,$$

where n is the normal inner unit vector of ∂Y. Choosing a local orthonormal frame $\{v, w = n \times v\}$ of $T\partial Y$, and using the fact that $D\psi = 0$, this is equal to

$$\int_{\partial Y} < w \times \nabla_v^\perp \psi - v \times \nabla_w^\perp \psi, \psi > d\sigma = -\int_{\partial Y} < \mathcal{D}_{\mu_X} \psi, \psi > d\sigma.$$

Summing up, we get the equation

$$\int_Y |\nabla^\perp \psi|^2 dy + \int_Y < \mathcal{R}_\nu \psi, \psi > dy + \int_{\partial Y} < \mathcal{D}_{\mu_X} \psi, \psi > d\sigma = 0. \tag{8}$$

Now we can prove the theorem 1.4. We see that if $\mathcal{D}_{\mu_X}$ and $\mathcal{R}_\nu$ are positive, then ψ vanishes. This means that our deformation problem has no cokernel, and by a straigthforward generalization of proposition 2.2, the moduli space is locally smooth. ■

3.3 Some properties of the operator $\mathcal{D}_L$

We sum up the main results about $\mathcal{D}_L$ in the following

Proposition 3.2 *Let Y an associative submanifold with boundary in a coassociative submanifold X, L a subbundle of ν over ∂Y, and $\mathcal{D}_L$ as defined in the introduction. Then $\mathcal{D}_L$ is of order 0, symmetric, and its trace is $2H$, where H is the mean curvature of ∂Y in Y with respect to the outside normal vector $-n$.*

Proof. Let L is a subbundle of ν invariant under the action of $n\times$. It is straighforward to check that $\mathcal{D}$ does not depend of the orthonormal frame $\{v, w = n \times v\}$. For every $\psi \in \Gamma(\partial Y, L)$ and f a function,

$$\begin{aligned} \mathcal{D}_L(f\psi) &= \pi_L(v \times \nabla_w(f\psi) - w \times \nabla_v(f\psi)) \\ &= f\mathcal{D}_L\psi + (d_w f)\pi_L(v \times \psi) - (d_v f)\pi_L(w \times \psi) = f\mathcal{D}_L\psi \end{aligned}$$

because $w \times L$ ans $v \times L$ are orthogonal to L. Now, decompose

$$\nabla^\top = \nabla^{\top\partial} + \nabla^{\perp\partial}$$

into its two projections along $T\partial Y$ and along the normal (in TY) n-direction. For the computations, choose v and $w = n \times v$ the two orthogonal characteristic directions on $T\partial Y$, i.e $\nabla_v^{\top\partial} n = -k_v v$ and $\nabla_w^{\top\partial} n = -k_w w$, where k_v and k_w are the two principal curvatures. We have $\nabla_v^{\perp\partial} = k_v n$ and $< \nabla_w^{\perp\partial} v, n >= 0$, and the same, *mutatis mutandis*, for w. Then, for ψ and $\phi \in \Gamma(\partial Y, L)$, using the fact that $T\partial Y \times L$ is orthogonal to L,

$$\begin{aligned} < \mathcal{D}_L \psi, \phi > &= < \nabla_w^\perp (v \times \psi) - (\nabla_w^{\perp\partial} v) \times \psi - \nabla_v^\perp (w \times \psi) + (\nabla_v^{\perp\partial} w) \times \psi, \phi > \\ &= < \nabla_w^\perp (v \times \psi) - \nabla_v^\perp (w \times \psi), \phi > \\ &= - < v \times \psi, \nabla_w^\perp \phi > + < w \times \psi, \nabla_v^\perp \phi > \\ &= < \psi, v \times \nabla_w^\perp \phi - w \times \nabla_v^\perp \phi >=< \psi, \mathcal{D}_L \phi > . \end{aligned}$$

To prove that the trace of $\mathcal{D}_L$ is $2H$, let $e \in L$ a local unit section of L. We have $n \times e \in L$ too, and

$$\begin{aligned} < \mathcal{D}_L(n \times e), n \times e > &= < v \times ((\nabla_w^{\top\partial} n) \times e) + v \times (n \times \nabla_w^\perp e), n \times e > \\ &\quad - < w \times (\nabla_v^{\top\partial} n) \times e - w \times (n \times \nabla_v^\perp e), n \times e > \\ &= < v \times (-k_w w \times e) - w \times (-k_v v \times e), n \times e > \\ &\quad + < v \times (n \times \nabla_w^\perp e) - w \times (n \times \nabla_v^\perp e), n \times e > \\ &= k_w + k_v - < n \times (w \times (n \times \nabla_v^\perp e) - v \times (n \times \nabla_w^\perp e)), e > \\ &= 2H - < \mathcal{D}_L e, e > . \end{aligned}$$

This shows that trace $\mathcal{D}_L = 2H$. ∎

3.4 Flatland

In flat spaces, R vanishes, and so $\mathcal{R}_\nu = -\mathcal{A}$. Hence a priori theorem 1.5 does'nt apply. Nevertheless, we have the

Corollary 3.3 *Let Y a totally geodesic associative submanifold in a flat M, with boundary in a coassociative X. If $\mathcal{D}_{\mu_X}$ positive, then $\mathcal{M}_{Y,X}$ is locally smooth and of expected dimension.*

Proof. The hypothesis on Y implies that $\mathcal{R}_\nu = 0$. Formula (8) shows that $\nabla^\perp \psi = 0$ and $\psi_{|\partial Y} = 0$. Using $d|\psi|^2 = 2 < \nabla^\perp \psi, \psi >= 0$, we get that $\psi = 0$, and $\mathrm{coker}(Y, \nu_X) = \ker(Y, \mu_X) = 0$. ∎

Proof of corollary 1.6. Let Y in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$, and $e \in \{0\} \times \mathbb{R}^4$. From [6] the boundary of Y lies in a local coassociatif submanifold X_e of $\mathbb{R}^7$, which contains $\partial Y \times \mathbb{R}e$ and whose tangent space over ∂Y is $T\partial Y \oplus \mathbb{R}e \oplus \mathbb{R}n \times e$. We see that Y has a direction of associative deformation along the fixed direction e, hence the dimension of the kernel of our problem is bigger than 1. On the other hand, the index is $c_1(\nu_X) + 1 - g = 1$. We want to show that $\mathcal{D}_{\mu_X}$ is positive. To see that, we choose orthogonal characteristic directions v and $w = n \times v$ in $T\partial Y$ as before. From theorem 1.3, we know that $v \times e$ is a non vanishing section of μ_X. We compute :

$$\begin{aligned} \mathcal{D}_{\mu_X}(v \times e) &= v \times (\nabla_w^{\perp\partial} v \times e) - w \times (\nabla_v^{\perp\partial} v \times e) \\ &= -k_v w \times (n \times e) = k_v v \times e. \end{aligned}$$

This shows that k_v is an eigenvalue of $\mathcal{D}_{\mu_X}$, and since we know that its trace is $2H$, we get that the other eigenvalue is k_w. Those eigenvalues are positive if the boundary of Y is strictly convex. By the last corollary, we get the result. ∎

Remark. In fact, we can give a better statement. Indeed, let $\psi \in \ker(D, \nu_X)$, and decompose $\psi_{|\partial Y}$ as $\psi = \psi_1 e + \psi_2 n \times e$. Of course, e is in the kernel of $\mathcal{D}_{\nu_X}$, and hence by proposition 1.4, the second term is an eigenvector of $\mathcal{D}_{\nu_X}$ for the eigenvalue $2H$. So formula (8) gives

$$\int_Y |\nabla^\perp \psi|^2 + \int_{\partial Y} 2H|\psi_2|^2 = 0.$$

If $H > 0$, this imply immediatly that $\psi_2 = 0$, and ψ_1 is constant, so ψ is proportional to e. This proves that $\dim \ker(D, \nu_X) = 1$ under the weaker condition that $H > 0$.

4 Extensions from the Calabi-Yau world

Closed extension. Let (N, J, Ω, ω) a 6-dimensional manifold with holonomy in $SU(3)$. Then $M = N \times S^1$ is a manifold with holonomy in G_2, with the calibration the 3-form given by $\phi = \omega \wedge dt + \Re\Omega$. Let L a special lagrangian 3-dimensional submanifold in N. Recall that since L is lagrangian, its normal bundle is simply JTL. Then $Y = L \times \{pt\}$ is an associative submanifold of $N \times S^1$, and its normal bundle ν is isomorphic to $JTL \times \mathbb{R}\partial_t$, where ∂_t is the dual vector field of dt. Since the translation along S^1 preserves the associativity of Y, we hence have $\mathcal{M}_L \times S^1 \subset \mathcal{M}_Y$. We prove that in fact, there is equality, without any condition (compare an equivalent result for coassociative submanifolds by Leung in [8]) :

Theorem 4.1 *The moduli space $\mathcal{M}_{L\times\{pt\}}$ of associative deformations of $L \times \{pt\}$ is always smooth, and can be identified with the product $\mathcal{M}_L \times S^1$.*

Proof. In this situation, we don't use the former expression of D^2. Instead, we give another formula for it. If $s = J\sigma \oplus \tau\partial_t$ is a section of ν, with $\sigma \in \Gamma(L, TL)$ and $\tau \in \Gamma(L, \mathbb{R}) = \Omega^0(L)$, we call $\sigma^\vee \in \Omega^1(L, \mathbb{R})$ the 1-form dual to σ, and we use the same symbol for its inverse. Moreover, we use the classical notation $* : \Omega^k(L) \to \Omega^{3-k}(L)$ for the Hodge star. Lastly, we define :

$$\begin{array}{rcl} D^\vee : \Omega^1(L) \times \Omega^0(L) & \longrightarrow & \Omega^1(L) \times \Omega^0(L) \\ (\alpha, \tau) & \mapsto & ((-J\pi_L D(J\alpha^\vee, \tau))^\vee, \pi_t D(J\alpha^\vee, \tau)), \end{array}$$

where π_L (resp. π_t) is the orthogonal projection $\nu = NL \oplus \mathbb{R}$ on the first (resp. the second) component. This is just a way to use forms on L instead of normal ambient vector fields.

Proposition 4.2 *For every $(\alpha, \tau) \in \Omega^1(L) \times \Omega^0(L)$,*

$$\begin{array}{rcl} D^\vee(\alpha, \tau) & = & (- * d\alpha - d\tau, *d * \alpha) \text{ and} \\ D^{\vee 2}(\alpha, \tau) & = & -\Delta(\alpha, \tau), \end{array}$$

where $\Delta = d^*d + dd^*$ (note that it is d^*d on τ).

Assuming for a while this propositioin, we see that for an infinitesimal associative deformation of $L \times \{pt\}$, then α and τ are harmonic over the compact L. In particular, τ is constant and α describes an infinitesimal special lagrangian deformation of L (see [10]). In other words, the only way to displace Y is to perturb L as special Lagrangian in N and translate it along the S^1-direction. Lastly, $\dim \mathrm{coker} D = \dim \ker D = b^1(L) + 1$ and by proposition 2.2, $\mathcal{M}_Y$ is smooth and of dimension $b^1(L) + 1$. ■

Proof of proposition 4.2. We will use the simple formula $\nabla^\perp Js = J\nabla^\top s$ for all sections $s \in \Gamma(L, NL)$. For $(s,\tau) \in \Gamma(L,NL)\times\mathbb{R}$, and e_i local orthonormal frame on L,

$$\begin{aligned}
D(s,\tau) &= \sum_{i,j} < e_i \times \nabla_i^\perp s, Je_j > Je_j + \sum_i < e_i \times \nabla_i^\perp s, \partial_t > \partial_t + \sum_i \partial_i\tau\ e_i \times \partial_t \\
&= J\sum_{i,j} \phi(e_i, \nabla_i^\perp s, Je_j)e_j + \sum_i \phi(e_i, \nabla_i^\perp s, \partial_t)\partial_t + J\sum_{i,j}\partial_i\tau \ \ < e_i\times\partial_t, Je_j > e_j,
\end{aligned}$$

where we used that $e_i \times \partial_t \perp \partial_t$.

$$\begin{aligned}
&= J\sum_{i,j} \Re\Omega(e_i, \nabla_i^\perp s, Je_j)e_j + \sum_i \omega(e_i, \nabla_i^\perp s)\partial_t + J\sum_{i,j}\partial_i\tau\ \phi(e_i,\partial_t, Je_j)e_j \\
&= J\sum_{i,j} \Re\Omega(e_i, J\nabla_i^\top \sigma, Je_j)e_j + \sum_i \omega(e_i, J\nabla_i^\top \sigma)\partial_t + J\sum_{i,j}\partial_i\tau\ \omega(Je_j, e_i)e_j,
\end{aligned}$$

where $\sigma = -Js \in \Gamma(L, TL)$.

$$\begin{aligned}
&= -J\sum_{i,j} \Re\Omega(e_i, \nabla_i^\top \sigma, e_j)e_j + \sum_i < e_i, \nabla_i^\top\sigma > \partial_t - J\sum_{i,j}\partial_i\tau < e_j, e_i > e_j \\
&= -J\sum_{i,j} Vol(e_i, \nabla_i^\top \sigma, e_j)e_j + \sum_i < e_i, \nabla_i^\top\sigma > \partial_t - J\sum_{i}\partial_i\tau e_i,
\end{aligned}$$

since $\Re\Omega$ is the volume form on TL. It is easy to find that this is equivalent to

$$D(s,\tau) = -J(*d\sigma^\vee)^\vee + (*d*\sigma^\vee)\partial_t - J(d\tau)^\vee,$$

and so

$$D^\vee(\sigma^\vee,\tau) = (-*d\sigma^\vee - d\tau, *d*\sigma^\vee).$$

Now, since $d^* = (-1)^{3p+1} * d*$ on the p-forms, one easy checks the formula for D^2. ■

Proof of theorem 1.7. Consider L a special lagrangian with boundary in a complex surface Σ, and $Y = L\times\{pt\}$ (resp. $X = \Sigma\times\{pt\}$) its associative (resp. coassociative) extension. It is clear that ν_X is equal (as a real bundle) to $JT\partial L$, and μ_X it the trivial $n\times$-bundle generated by ∂_t. We begin by computing the index of the boundary problem. This is very easy, since μ_X is trivial, and by theorem 1.3, we have $\nu_X \cong T\partial L^*$ (as $n\times$-bundles. Hence the index is

$$-c_1(T\partial L) + 1 - g = -(2-g) + 1 - g = g - 1,$$

where g is the genus of ∂Y. Now let $\psi = s + \tau\frac{\partial}{\partial t}$ belonging to $\mathrm{coker}(D,\nu_X) = \ker(D,\mu_X)$, where s a section of NL and $\tau\in\Gamma(L,\mathbb{R})$. Let $\alpha = -Js^\vee$. By proposition 4.2, α is a harmonic 1-form, and τ is harmonic (note that Y is note closed, so τ may be not constant). By classical results for harmonic 1-forms, we have :

$$\frac{1}{2}\Delta|\psi|^2 = \frac{1}{2}\Delta(|\alpha|^2 + |\tau|^2) = |\nabla_L\alpha|^2 + |d\tau|^2 + \frac{1}{2}\mathrm{Ric}\ (\alpha,\alpha).$$

Integrating on $L\times\{pt\}$, we obtain the equivalence of formula (8) :

$$-\int_{\partial Y} < \mathcal{D}_{\mu_X}\psi, \psi > d\sigma = \int_Y |\nabla_L\alpha|^2 + |d\tau|^2 + \frac{1}{2}\mathrm{Ric}\ (\alpha,\alpha)dy.$$

Lastly, let us compute the eigenvalues of $\mathcal{D}_{\mu_X}$. The constant vector $\frac{\partial}{\partial t}$ over ∂Y lies clearly in the kernel of $\mathcal{D}_{\mu_X}$. By proposition 3.2, the other eigenvalue of $\mathcal{D}_{\mu_X}$ is $2H$, with eigenspace generated by $n \times \frac{\partial}{\partial t}$. Over ∂Y, s lies in $JTL \cap \mu_X$, hence is proportional to $n \times \frac{\partial}{\partial t}$. Consequently, $\mathcal{D}_{\mu_X}\psi = 2Hs$ and

$$-\int_{\partial Y} 2H|s|^2 d\sigma = \int_Y |\nabla_L \alpha|^2 + |d\tau|^2 + \frac{1}{2}\text{Ric }(\alpha,\alpha)dy.$$

This equation, the positivity of the Ricci curvature and the positivity of H show that α vanishes and τ is constant. So we see that $\dim \operatorname{coker}(Y,X) = 1$, and by the constant rank theorem, $\mathcal{M}_{Y,X}$ is locally smooth and of dimension $\dim \ker(Y,X) = g$. ■

5 Computation of D^2

Proof of theorem 1.2. Before diving into the calculi, we need the following trivial lemma :

Lemma 5.1 *Let ∇ the Levi-Civita connection on M and R its curvature tensor. For any vector fields w, z, u and v on M, we have*

$$\begin{aligned} \nabla(u \times v) &= \nabla u \times v + u \times \nabla v \textit{ and} \\ R(w,z)(u \times v) &= R(w,z)u \times v + u \times R(w,z)v. \end{aligned}$$

If Y is an associative submanifold of M with normal bundle ν, $u \in \Gamma(Y,TY)$, $v \in \Gamma(Y,TY)$ and $\eta \in \Gamma(Y,\nu)$, then

$$\begin{aligned} \nabla^\top(u \times v) &= \nabla^\top u \times v + u \times \nabla^\top v \textit{ and} \\ \nabla^\perp(u \times \eta) &= \nabla^\top u \times v + u \times \nabla^\perp v, \end{aligned}$$

where $\nabla^\top = \nabla - \nabla^\perp$ is the orthogonal projection of ∇ on TY.

Proof. Let $x_1, \cdots, x_7$ normal coordinates on M near x, and $e_i = \frac{\partial}{\partial x_i}$ their derivatives, orthonormal at x. We have

$$u \times v = \sum_i < u \times v, e_i > e_i = \sum_i \phi(u,v,e_i)e_i,$$

so that at x, where $\nabla_{e_j} e_i = 0$,

$$\begin{aligned} \nabla(u \times v) &= \sum_i (\nabla\phi(u,v,e_i) + \phi(\nabla u, v, e_i) + \phi(u, \nabla v, e_i) + \phi(\nabla u, v, \nabla e_i))e_i \\ &= \sum_i (\phi(\nabla u, v, e_i) + \phi(u, \nabla v, e_i))e_i = \nabla u \times v + u \times \nabla v, \end{aligned}$$

because $\nabla\phi = 0$. Now if u and v are in TY, then we get the result after remarking that $(\nabla u \times v)^\top = \nabla^\top u \times v$, because TY is invariant under $\times$. The last relation is implied by $TY \times \nu \subset \nu$ and $\nu \times \nu \subset TY$. The curvature relation is easily derived from the definition $R(w,z) = \nabla_w \nabla_z - \nabla_z \nabla_w - \nabla_{[w,z]}$ and the derivation of the vector product. ■

We compute D^2 at a point $x \in Y$. For this, we choose normal coordinates on Y and $e_i \in \Gamma(Y, TY)$ their associated derivatives, orthonormal at x. To be explicit, $\nabla^\top e_i = 0$ at x. Let $\psi \in \Gamma(Y, \nu)$.

$$\begin{aligned}
D^2\psi &= \sum_{i,j} e_i \times \nabla_i^\perp (e_j \times \nabla_j^\perp \psi) \\
&= \sum_{i,j} e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \psi) + \sum_{i,j} e_i \times (\nabla_i^\top e_j \times \nabla_j^\perp \psi) \\
&= -\sum_i \nabla_i^\perp \nabla_i^\perp \psi - \sum_{i \neq j} (e_i \times e_j) \times \nabla_i^\perp \nabla_j^\perp \psi \\
&= \nabla^{\perp *} \nabla^\perp \psi - \sum_{i<j} (e_i \times e_j) \times (\nabla_i^\perp \nabla_j^\perp - \nabla_j^\perp \nabla_i^\perp)\psi \\
&= \nabla^{\perp *} \nabla^\perp \psi - \sum_{i<j} (e_i \times e_j) \times R^\perp(e_i, e_j)\psi.
\end{aligned}$$

Since $(e_i \times e_j) \times R^\perp(e_i, e_j)$ is symmetric in i, j, this is equal to

$$\nabla^{\perp *} \nabla^\perp \psi - \frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R^\perp(e_i, e_j)\psi.$$

The main tool for the sequence is the Ricci equation. Let u, v in $\Gamma(Y, TY)$ and ϕ, ψ in $\Gamma(Y, \nu)$.

$$< R^\perp(u, v)\psi, \phi > = < R(u, v)\psi, \phi > + < (A_\psi A_\phi - A_\phi A_\psi)u, v >,$$

where $A_\phi(u) = A(\phi)(u) = -\nabla_u^\top \phi$. Choosing $\eta_1, \cdots, \eta_4$ an orthonormal basis of ν at the point x, we get

$$\begin{aligned}
-\frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R^\perp(e_i, e_j)\psi &= -\frac{1}{2} \sum_{i,j,k} < (e_i \times e_j) \times R^\perp(e_i, e_j)\psi, \eta_k > \eta_k \\
&= \frac{1}{2} \sum_{i,j,k} < R^\perp(e_i, e_j)\psi, (e_i \times e_j) \times \eta_k > \eta_k \\
&= -\frac{1}{2} \pi_\nu \sum_{i,j} (e_i \times e_j) \times R(e_i, e_j)\psi \\
&\quad + \frac{1}{2} \sum_{i,j,k} < (A_\psi A_{(e_i \times e_j) \times \eta_k} - A_{(e_i \times e_j) \times \eta_k} A_\psi) e_i, e_j > \eta_k.
\end{aligned}$$

Using the classical Bianchi relation $R(e_i, e_j)\psi = -R(\psi, e_i)e_j - R(e_j, \psi)e_i$, the first part of the sum is equal to

$$\begin{aligned}
I = -2\pi_\nu (e_1 \times R(e_2, \psi)e_3 + e_2 \times R(e_3, \psi)e_1 + e_3 \times R(e_1, \psi)e_2) = \\
-2\pi_\nu (e_1 \times R(e_2, \psi)(e_1 \times e_2) + e_2 \times R(e_3, \psi)(e_2 \times e_3) + e_3 \times R(e_1, \psi)(e_3 \times e_1)) = \\
-2\pi_\nu (e_1 \times (R(e_2, \psi)e_1 \times e_2 + e_1 \times R(e_2, \psi)e_2) + e_2 \times (R(e_3, \psi)e_2 \times e_3 + e_2 \times R(e_3, \psi)e_1) + \\
e_3 \times (R(e_1, \psi)e_3 \times e_1 + e_3 \times R(e_1, \psi)e_2)) = \\
-I + 2\pi_\nu \sum_i R(e_i, \psi)e_i,
\end{aligned}$$

which gives $I = \pi_\nu \sum_i R(e_i, \psi)e_i$.

The Weingarten endomorphisms are symmetric, so that the second part of the sum is

$$\frac{1}{2}\sum_{i,j,k} < A_{(e_i\times e_j)\times\eta_k} e_i, A_\psi e_j > \eta_k - \frac{1}{2}\sum_{i,j,k} < A_\psi e_i, A_{(e_i\times e_j)\times\eta_k} e_j > \eta_k.$$

It is easy to see that the second sum is the opposite of the first one. We compute

$$A_{(e_i\times e_j)\times\eta_k} e_i \quad = \quad -(\nabla_i^\perp e_i \times e_j)\times\eta_k - (e_i\times\nabla_i^\perp e_j)\times\eta_k + (e_i\times e_j)\times A_{\eta_k} e_i.$$

But we know that an associative submanifold is minimal, so that

$$\sum_i \nabla_i^\perp e_i = 0.$$

Moreover, deriving the relation $e_3 = \pm e_1 \times e_2$, one easily check that

$$\sum_i e_i \times \nabla_j^\perp e_i = 0.$$

Summing, the only resting term is

$$\sum_{i,j,k} < (e_i\times e_j)\times A_{\eta_k} e_i, A_\psi e_j > \eta_k.$$

We now use the classical formula for vectors u, v and w in TY :

$$(v\times w)\times u =< u, v > w - < u, w > v,$$

hence

$$(e_i\times e_j)\times A_{\eta_k} e_i =< A_{\eta_k} e_i, e_i > e_j - < A_{\eta_k} e_i, e_j > e_i.$$

One more simplification comes from $\sum_i < A_{\eta_k} e_i, e_i >= 0$ for all k because since Y is minimal, so our sum is now equal to

$$-\sum_{i,j,k} < A_{\eta_k} e_i, e_j >< e_i, A_\psi e_j > \eta_k = -\mathcal{A}\psi.$$

∎

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